In this note, we show that the proof of [EGA Théorème III.3.2.1] can be slightly modified to avoid spectral sequences. The statement of the theorem is as follows:

Let $Y$ be a locally Noetherian scheme and $f : X \to Y$ a proper morphism. For each coherent $\mathcal{O}_X$-module $\mathcal{F}$, the $\mathcal{O}_Y$-modules $R^q f_*(\mathcal{F})$ are coherent for $q \geq 0$.

The previous results [EGA Théorème III.3.1.2, Corollaire III.3.1.3] reduce the problem to show the following fact:

1. For each irreducible closed subset $Z$ of $X$, with generic point $z$, there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}_z \neq 0$ and $R^q f_*(\mathcal{F})$ is coherent for $q \geq 0$.

To prove (1), we can suppose that $Z = X$ is integral, i.e., it suffices to prove:

2. If $X$ is integral with generic point $x$, there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}_x \neq 0$ and $R^q f_*(\mathcal{F})$ is coherent for $q \geq 0$.

To prove that (2) $\Rightarrow$ (1), we consider $Z$ as reduced (and hence integral) closed subscheme of $X$ and take the corresponding closed immersion $j : Z \to X$. Since $f \circ j$ is a proper morphism, we know that there exists a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that $\mathcal{G}_z \neq 0$ and $R^q (f \circ j)_* (\mathcal{G})$ is coherent for $q \geq 0$. Now, the $\mathcal{O}_X$-module $\mathcal{F} = j_* (\mathcal{G})$ is coherent by [L, 5.1.14 d] and it satisfies (1). The fact that $R^q f_*(\mathcal{F})$ is coherent is a consequence of the equality $R^q (f \circ j)_* (\mathcal{G}) = R^q f_* (j_* (\mathcal{G}))$, which can be proved by using that $j_*$ is an exact functor and the same argument we use for proving (3) below.

Now let us prove (2): By Chow’s lemma, there exists a projective and surjective morphism $g : X' \to X$ such that $f \circ g : X' \to Y$ is also projective. Let $\mathcal{O}_X(1)$ be a very ample sheaf on $X'$ with respect to $g$. By [H, III.8.8] there exists an integer $n \geq 1$ such that $\mathcal{F} = g_*(\mathcal{O}_X(n))$ is a coherent $\mathcal{O}_X$-module, the natural map $g^* g_*(\mathcal{O}_X(n)) \to \mathcal{O}_X(n)$ is surjective and $R^q g_*(\mathcal{O}_X(n)) = 0$ for $q \geq 1$.

The surjectivity of $g^* \mathcal{F} \to \mathcal{F}$ implies that $\mathcal{F}_x \neq 0$. Now it suffices to prove that

$$R^q f_*(\mathcal{F}) = R^q (f \circ g)_* (\mathcal{O}_X(1)),$$

since the $\mathcal{O}_Y$-modules on the right are coherent by [H, III.8.8]. (This is the only step of the proof where spectral sequences are used in [EGA].)

We start from an injective resolution of $\mathcal{O}_X(1)$. By definition of derived functors, the fact that $R^q g_*(\mathcal{O}_X(1)) = 0$ means that the sequence remains exact if we apply to it the functor $g_*$ and so, we get a resolution of $\mathcal{F}$, which is obviously flasque. Hence, it can be used to calculate the right hand side of (3). (See [H, III.8.3 and III.1.2A].) So, if we apply the functor $f_*$ and take the cohomology groups, we are calculating both sides of (3), and this proves (2).

References

